J Glob Optim (2010) 46:319–329
DOI 10.1007/s10898-009-9427-x

Strong convergence theorems for variational inequalities and relatively weak nonexpansive mappings

Ying Liu

Received: 2 December 2008 / Accepted: 23 April 2009 / Published online: 9 May 2009

Abstract In this paper, we introduce an iterative sequence for finding a common element of the set of fixed points of a relatively weak nonexpansive mapping and the set of solutions of a variational inequality in a Banach space. Our results extend and improve the recent ones announced by Li (J Math Anal Appl 295:115–126, 2004), Jianghua (J Math Anal Appl 337:1041–1047, 2008), and many others.

Keywords Variational inequalities · Relatively weak nonexpansive mappings · Generalized projection · Cauchy sequences · Continuity

Mathematics Subject Classification (2000) 47H09 · 47H05 · 47J25 · 47J05

1 Introduction

Let B be a Banach space, B^* be the dual space of B. $\langle \cdot, \cdot \rangle$ denotes the duality pairing of B^* and B. Let K be a nonempty closed convex subset of B and $T : K \to B^*$ be an operator. We consider the following variational inequality:

$$\text{Find } x \in K, \text{ such that } \langle Tx, y - x \rangle \geq 0, \quad \text{for all } y \in K. \tag{1.1}$$

A point $x_0 \in K$ is called a solution of the variational inequality (1.1) if for every $y \in K$, $\langle Tx_0, y - x_0 \rangle \geq 0$. The set of solutions of the variational inequality (1.1) is denoted by $VI(K, T)$. The variational inequality (1.1) has been intensively considered due to its various applications in operations research, economic equilibrium and engineering design. When T has some monotonicity, many iterative methods for solving the variational inequality (1.1) have been developed, e.g., see [1–7].

Y. Liu (✉)
College of Mathematics and Computer, Hebei University, 071002 Baoding, People's Republic of China
e-mail: ly_cyh2007@yahoo.com.cn

Most recently, applying the generalized projection operator in uniformly convex and uniformly smooth Banach spaces, Li [8] established the following convergence theorem of Mann type iterative scheme for variational inequalities without assuming the monotonicity of T in compact subsets of Banach spaces:

Theorem K1 (Li [8], Theorem 3.1) *Let B be a uniformly convex and uniformly smooth Banach space and let K be a compact convex subset of B. Let $T : K \to B^*$ be a continuous mapping on K such that*

$$\langle Tx - \xi, J^*(Jx - (Tx - \xi))\rangle \geq 0, \quad \text{for all } x \in K,$$

where $\xi \in B^*$. For any $x_0 \in K$, define a Mann type iteration scheme as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \pi_K(Jx_n - (Tx_n - \xi)), \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\}$ satisfies conditions

$$\text{(a) } 0 \leq \alpha_n \leq 1, \quad \text{for all } n \in N; \text{ (b) } \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty.$$

Then the variational inequality $\langle Tx - \xi, y - x\rangle \geq 0, \forall y \in K$, [when $\xi = 0$, this is the variational inequality (1.1)] has a solution $x^* \in K$ and there exists a subsequence $\{n_i\} \subset \{n\}$ such that

$$x_{n_i} \to x^*, \quad \text{as } i \to \infty.$$

In addition, Fan [9] established some existence results of solutions and the convergence of a Mann type iterative scheme for the variational inequality (1.1) in noncompact subsets of Banach spaces. He proved the following theorem:

Theorem K2 (Fan [9], Theorem 3.3) *Let B be a uniformly convex and uniformly smooth Banach space and let K be a closed convex subset of B. Suppose that there exists a positive number β, such that*

$$\langle Tx, J^*(Jx - \beta Tx)\rangle \geq 0, \quad \text{for all } x \in K,$$

and $J - \beta T : K \to B^*$ is compact. If

$$\langle Tx, y\rangle \leq 0, \quad \text{for all } x \in K, y \in VI(K, T),$$

then the variational inequality (1.1) has a solution $x^* \in K$ and the sequence $\{x_n\}$ defined by the following iteration scheme:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \pi_K(Jx_n - \beta Tx_n), \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\}$ satisfies: $0 < a \leq \alpha_n \leq b < 1$ for all $n \in N$, for some positive numbers $a, b \in (0, 1)$ satisfying $a < b$, converges strongly to $x^* \in K$.

On the other hand, Kohasaka and Takahashi [10] introduced the definition of the relatively weak nonexpansive mapping. They proved that $J_r = (J + rA)^{-1}J$, for $r > 0$ is relatively weak nonexpansive, where $A \subset B \times B^*$ is a continuous monotone mapping with $A^{-1}0 \neq \emptyset$ and B is a smooth, strictly convex and reflexive Banach space.

Motivated by these facts, our purpose in this paper is to establish an iteration sequence for approximating a common element of the set of fixed points of a relatively weak nonexpansive mapping and the set of solutions of the variational inequality (1.1) in noncompact subsets of Banach spaces without assuming the compactness of the operator $J - \beta T$.

2 Preliminaries

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively.

Let X, Y be Banach spaces, $T : D(T) \subset X \to Y$, the operator T is said to be compact if it is continuous and maps the bounded subsets of $D(T)$ onto the relatively compact subsets of Y.

We denote by $J : B \to 2^{B^*}$ the normalized duality mapping from B to 2^{B^*}, defined by

$$J(x) := \{v \in B^* : \langle v, x\rangle = \|v\|^2 = \|x\|^2\}, \quad \forall x \in B.$$

The duality mapping J has the following properties:

(i) if B is smooth, then J is single-valued;
(ii) if B is strictly convex, then J is one-to-one;
(iii) if B is reflexive, then J is surjective.
(iv) if B is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of B.

Let B be a reflexive, strictly convex, smooth Banach space and J the duality mapping from B into B^*. Then J^* is also single-valued, one-to-one, surjective, and it is the duality mapping from B^* into B, i.e. $J^*J = I$.

When $\{x_n\}$ is a sequence in B, we denote strong convergence of $\{x_n\}$ to $x \in B$ by $x_n \to x$.

Let $U = \{x \in B : \|x\| = 1\}$. A Banach space B is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in U$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in U and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. A Banach space B is said to be smooth provided $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$.

In [2,4], Alber introduced the functional $V : B^* \times B \to R$ defined by

$$V(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2,$$

where $\phi \in B^*$ and $x \in B$.

It is easy to see that

$$V(\phi, x) \geq (\|\phi\| - \|x\|)^2. \tag{2.1}$$

Thus the functional $V : B^* \times B \to R^+$ is nonnegative.

Definition 2.1 (See [9]) If B is a uniformly convex and uniformly smooth Banach space, the generalized projection $\pi_K : B^* \to K$ is a mapping that assigns an arbitrary point $\phi \in B^*$ to the minimum point of the functional $V(\phi, x)$, i.e., a solution to the minimization problem

$$V(\phi, \pi_K(\phi)) = \inf_{y\in K} V(\phi, y).$$

Li [11] proved that the generalized projection operator $\pi_K : B^* \to K$ is continuous, if B is a reflexive, strictly convex and smooth Banach space.

The functional $\phi : B \times B \to R$ is defined by

$$\phi(x, y) = V(Jy, x), \quad \forall x, y \in B.$$

The following properties of the operators π_K, V are useful for our paper. (See, for example, [1,11])

(i) $V : B^* \times B \to R$ is continuous.
(ii) $V(\phi, x) = 0$ if and only if $\phi = Jx$.
(iii) $V(J\pi_K\phi, x) \le V(\phi, x)$ for all $\phi \in B^*$ and $x \in B$.
(iv) The operator π_K is J fixed at each point $x \in K$, i.e., $\pi_K(Jx) = x$.
(v) If B is smooth, then for any given $\phi \in B^*, x \in K, x \in \pi_K\phi$ if and only if $\langle \phi - Jx, x - y\rangle \ge 0$, for all $y \in K$.
(vi) The operator $\pi_K : B^* \to K$ is single valued if and only if B is strictly convex.
(vii) If B is smooth, then for any given point $\phi \in B^*, x \in \pi_K\phi$, the following inequality holds

$$V(Jx, y) \le V(\phi, y) - V(\phi, x) \quad \forall y \in K.$$

(viii) $V(\phi, x)$ is convex with respect to ϕ when x is fixed and with respect to x when ϕ is fixed.
(ix) If B is reflexive, then for any point $\phi \in B^*$, $\pi_K(\phi)$ is a nonempty, closed, convex and bounded subset of K.

Remark 2.1 It is easy to see that if B is a strictly convex and smooth Banach space, then for $x, y \in B, \phi(x, y) = 0$, i.e.$V(Jy, x) = 0$ if and only if $x = y$. It is sufficient to show that if $V(Jy, x) = 0$ then $x = y$. From property (ii) of the operator V, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$.

Using the properties of generalized projection operator π_K, Alber proved the following theorem in [1].

Theorem 2.1 *Let B be a reflexive, strictly convex and smooth Banach space with dual space B^*. Let T be an arbitrary operator from Banach space B to B^*, α an arbitrary fixed positive number. Then the point $x \in K \subset B$ is a solution of variational inequality (1.1) if and only if x is a solution of the operator equation in B*

$$x = \pi_K(Jx - \alpha Tx).$$

Let S be a mapping from K into itself. We denote by $F(S)$ the set of fixed points of S. A point p in K is said to be an asymptotic fixed point of S [12] if K contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of asymptotic fixed point of S will be denoted by $\hat{F}(S)$. A mapping S from K into itself is called relatively nonxpansive (see e.g., [12]) if $\hat{F}(S) = F(S)$ and $\phi(p, Sx) \le \phi(p, x)$ for all $x \in K$ and $p \in F(S)$. The asymptotic behavior of relatively nonexpansive mappings were studied in [12,13]. A point p in K is said to be a strong asymptotic fixed point of S if K contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of strong asymptotic fixed points of S will be denoted by $\tilde{F}(S)$. A mapping S from K into itself is called relatively weak nonexpansive if $\tilde{F}(S) = F(S)$ and $\phi(p, Sx) \le \phi(p, x)$ for all $x \in K$ and $p \in F(S)$. If B is a smooth strictly convex and reflexive Banach space, and $A \subset B \times B^*$ is a continuous monotone mapping with $A^{-1}0 \ne \emptyset$, then it is proved in [10] that $J_r = (J + rA)^{-1}J$, for $r > 0$ is relatively weak nonexpansive. Moreover, if $S : K \to K$ is relatively weak nonexpansive, then using the definition of ϕ (i.e. the same argument as in the proof of [14, p.260]) one can show that $F(S)$ is closed and convex.

It is obvious that relatively nonexpansive mapping is relatively weak nonexpansive mapping. In fact, for any mapping $S : K \to K$ we have $F(S) \subset \tilde{F}(S) \subset \hat{F}(S)$. Therefore, if S is a relatively nonexpansive mapping, then $F(S) = \tilde{F}(S) = \hat{F}(S)$.

The following lemmas are useful for the proof of our main theorem.

Lemma 2.2 (See [14]) *Let B be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of B. If $\phi(y_n, z_n) \to 0$, and either $\{y_n\}$, or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 2.3 (See [15]) *Let B be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

for all $x, y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in B : \|z\| \leq r\}$.

Lemma 2.4 (See [5]) *Let B be a uniformly convex and uniformly smooth Banach space. We have*

$$\|\phi + \Phi\|^2 \leq \|\phi\|^2 + 2\langle \Phi, J^*(\phi + \Phi)\rangle, \qquad \forall \phi, \Phi \in B^*.$$

Lemma 2.5 *Let B be a uniformly convex and uniformly smooth Banach space, let K be a nonempty, closed convex subset of B. Suppose that there exists a positive number β such that*

$$\langle Tx, J^*(Jx - \beta Tx)\rangle \geq 0, \quad \text{for all } x \in K \tag{2.2}$$

and

$$\langle Tx, y\rangle \leq 0, \quad \forall x \in K, y \in VI(K, T). \tag{2.3}$$

Then $VI(K, T)$ is closed and convex.

Proof We first show that $VI(K, T)$ is closed. Let $\{x_n\}$ be a sequence of $VI(K, T)$ such that $x_n \to \hat{x} \in K$. From the definition of ϕ, the property of V, lemma 2.4 and conditions (2.2) (2.3), we have

$$\begin{aligned}
\phi(x_n, \pi_K(J\hat{x} - \beta T\hat{x})) &= V(J\pi_K(J\hat{x} - \beta T\hat{x}), x_n) \\
&\leq V(J\hat{x} - \beta T\hat{x}, x_n) \\
&= \|J\hat{x} - \beta T\hat{x}\|^2 - 2\langle J\hat{x} - \beta T\hat{x}, x_n\rangle + \|x_n\|^2 \\
&\leq \|J\hat{x}\|^2 - 2\beta\langle T\hat{x}, J^*(J\hat{x} - \beta T\hat{x})\rangle - 2\langle J\hat{x}, x_n\rangle \\
&\quad + 2\beta\langle T\hat{x}, x_n\rangle + \|x_n\|^2 \\
&\leq \|J\hat{x}\|^2 - 2\langle J\hat{x}, x_n\rangle + \|x_n\|^2 \\
&= \phi(x_n, \hat{x}),
\end{aligned}$$

for each $n \in N$. This implies,

$$\begin{aligned}
0 \leq \phi(\hat{x}, \pi_K(J\hat{x} - \beta T\hat{x})) &= \lim_{n\to\infty} \phi(x_n, \pi_K(J\hat{x} - \beta T\hat{x})) \\
&\leq \lim_{n\to\infty} \phi(x_n, \hat{x}) = \phi(\hat{x}, \hat{x}) = 0.
\end{aligned}$$

Therefore, we obtain $\hat{x} = \pi_K(J\hat{x} - \beta T\hat{x})$. So,we have $\hat{x} \in VI(K, T)$. Next, we show that $VI(K, T)$ is convex. For $x, y \in VI(K, T)$, and $t \in (0, 1)$, put $z = tx + (1-t)y$. It is sufficient to show $z = \pi_K(Jz - \beta Tz)$. In fact, we have

$$\begin{aligned}
0 &\leq \phi(z, \pi_K(Jz - \beta Tz)) = V(J\pi_K(Jz - \beta Tz), z) \\
&\leq V(Jz - \beta Tz, z) = \|Jz - \beta Tz\|^2 - 2\langle Jz - \beta Tz, z\rangle + \|z\|^2 \\
&\leq \|Jz\|^2 - 2\beta\langle Tz, J^*(Jz - \beta Tz)\rangle - 2\langle Jz, z\rangle + 2\beta\langle Tz, z\rangle + \|z\|^2 \\
&= -2\beta\langle Tz, J^*(Jz - \beta Tz)\rangle + 2\beta\langle Tz, z\rangle.
\end{aligned} \tag{2.4}$$

By (2.4), (2.2) and (2.3), we have

$$\begin{aligned}0 &\le \phi(z, \pi_K(Jz-\beta Tz)) \le -2\beta\langle Tz, J^*(Jz-\beta Tz)\rangle + 2\beta\langle Tz, z\rangle \\ &\le 2\beta\langle Tz, z\rangle = 2\beta\langle Tz, tx+(1-t)y\rangle = 2\beta t\langle Tz, x\rangle + 2\beta(1-t)\langle Tz, y\rangle \le 0.\end{aligned}$$

This implies $z = \pi_K(Jz-\beta Tz)$. Therefore, $VI(K, T)$ is closed and convex. □

Lemma 2.6 *If B is a reflexive, strictly convex and smooth Banach space, then $\pi_B = J^*$.*

Proof For every $\phi \in B^*$, by definition of V and (2.1), we have

$$0 \le V(\phi, J^*\phi) = \|\phi\|^2 - 2\langle\phi, J^*\phi\rangle + \|J^*\phi\|^2 = 0.$$

By definition of the operator π_B, we have $J^*\phi \in \pi_B\phi$. Since π_B is single-valued, we have $\pi_B\phi = J^*\phi$. □

3 Main results

For any $x_0 \in K$, we define the iteration process $\{x_n\}$ as follows:

$$\begin{cases} x_0 \ \in \ K \ chosen \ arbitrarily, \\ z_n \ = \ \pi_K(\beta_n Jx_n + (1-\beta_n)JSx_n), \\ y_n \ = \ J^*(\alpha_n Jx_n + (1-\alpha_n)J\pi_K(Jz_n - \beta Tz_n)), \\ C_0 \ = \ \{u \in K : \phi(u, y_0) \le \phi(u, x_0)\}, \\ C_n \ = \ \{u \in C_{n-1} \bigcap Q_{n-1} : \phi(u, y_n) \le \phi(u, x_n)\}, \\ Q_0 \ = \ K, \\ Q_n \ = \ \{u \in Q_{n-1} \bigcap C_{n-1} : \langle Jx_0 - Jx_n, x_n - u\rangle \ge 0\}, \\ x_{n+1} \ = \ \pi_{C_n \bigcap Q_n} Jx_0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy:

$$0 \le \alpha_n < 1, \text{ and } \limsup_{n\to\infty} \alpha_n < 1; 0 < \beta_n < 1 \text{ and } \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Theorem 3.1 *Let B be a uniformly convex and uniformly smooth Banach space. Let K be a nonempty, closed convex subset of B. Assume that T is an operator of K into B^* that satisfy conditions (2.2) and (2.3) and $S : K \to K$ is a relatively weak nonexpansive mapping with $VI(K, T) \bigcap F(S) \ne \emptyset$. If $T : K \to B^*$ is continuous, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $\pi_{VI(K,T) \bigcap F(S)} Jx_0$.*

Proof We first show that C_n and Q_n are closed and convex for each $n \in N \bigcup \{0\}$. By the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \in N \bigcup \{0\}$. We show that C_n is convex. Since $\phi(u, y_n) \le \phi(u, x_n)$ is equivalent to

$$2\langle Jx_n - Jy_n, u\rangle + \|y_n\|^2 - \|x_n\|^2 \le 0,$$

it follows that C_n is convex. Next, we show that $VI(K, T) \bigcap F(S) \subset C_n \bigcap Q_n$ for all $n \in N \bigcup \{0\}$. Let $p \in VI(K, T) \bigcap F(S)$, then, from the definitions of ϕ and V, property (iii) of V, lemma 2.4, conditions (2.2) and (2.3), we have

$$\begin{aligned}\phi(p, \pi_K(Jz_n - \beta Tz_n)) &= V(J\pi_K(Jz_n - \beta Tz_n), p) \le V(Jz_n - \beta Tz_n, p) \\ &= \|Jz_n - \beta Tz_n\|^2 - 2\langle Jz_n - \beta Tz_n, p\rangle + \|p\|^2 \\ &\le \|Jz_n\|^2 - 2\beta\langle Tz_n, J^*(Jz_n - \beta Tz_n)\rangle \\ &\quad - 2\langle Jz_n - \beta Tz_n, p\rangle + \|p\|^2 \\ &\le \|Jz_n\|^2 - 2\langle Jz_n, p\rangle + \|p\|^2 = \phi(p, z_n),\end{aligned} \tag{3.2}$$

for each $n \in N \bigcup \{0\}$. Therefore, by properties (viii) and (iii) of the operator V, (3.2), the definition of S, we obtain

$$\begin{aligned}
\phi(p, y_0) &= V(Jy_0, p) \\
&\leq \alpha_0 V(Jx_0, p) + (1-\alpha_0)V(J\pi_K(Jz_0 - \beta Tz_0), p) \\
&= \alpha_0\phi(p, x_0) + (1-\alpha_0)\phi(p, \pi_K(Jz_0 - \beta Tz_0)) \\
&\leq \alpha_0\phi(p, x_0) + (1-\alpha_0)\phi(p, z_0) \\
&= \alpha_0\phi(p, x_0) + (1-\alpha_0)V(Jz_0, p) \\
&\leq \alpha_0\phi(p, x_0) + (1-\alpha_0)V(\beta_0 Jx_0 + (1-\beta_0)JSx_0, p) \\
&\leq \alpha_0\phi(p, x_0) + (1-\alpha_0)(\beta_0 V(Jx_0, p) + (1-\beta_0)V(JSx_0, p)) \\
&= \alpha_0\phi(p, x_0) + (1-\alpha_0)(\beta_0\phi(p, x_0) + (1-\beta_0)\phi(p, Sx_0)) \\
&\leq \alpha_0\phi(p, x_0) + (1-\alpha_0)(\beta_0\phi(p, x_0) + (1-\beta_0)\phi(p, x_0)) \\
&= \phi(p, x_0),
\end{aligned} \tag{3.3}$$

which gives that $p \in C_0$. On the other hand, it is clear that $p \in Q_0 = K$. Thus $VI(K, T) \bigcap F(S) \subset C_0 \bigcap Q_0$ and hence $x_1 = \pi_{C_0 \bigcap Q_0} Jx_0$ is well defined. Suppose that $VI(K, T) \bigcap F(S) \subset C_{n-1} \bigcap Q_{n-1}$ and x_n is well defined. Then the method in (3.3) implies that $\phi(p, y_n) \leq \phi(p, x_n)$ and that $p \in C_n$. Moreover, it follows from property(v) of the operator π_K and $x_n = \pi_{C_{n-1} \bigcap Q_{n-1}} Jx_0$ that

$$\langle Jx_0 - Jx_n, x_n - p \rangle \geq 0,$$

which implies that $p \in Q_n$. Hence $VI(K, T) \bigcap F(S) \subset C_n \bigcap Q_n$ and $x_{n+1} = \pi_{C_n \bigcap Q_n} Jx_0$ is well-defined. Then by induction, $VI(K, T) \bigcap F(S) \subset C_n \bigcap Q_n$ for each $n \in N \bigcup \{0\}$. Hence, the sequence $\{x_n\}$ generated by (3.1) is well defined.

It follows from the definition of Q_n that $x_n = \pi_{Q_n} Jx_0$. Using $x_n = \pi_{Q_n} Jx_0$ and $VI(K, T) \bigcap F(S) \subset Q_n$, we have $V(Jx_0, x_n) \leq V(Jx_0, p)$ for each $p \in VI(K, T) \bigcap F(S)$. Therefore, $\{V(Jx_0, x_n)\}$ is bounded. Moreover, from the definition of V, we have that $\{x_n\}$ is bounded. Since $x_{n+1} = \pi_{C_n \bigcap Q_n} Jx_0 \in Q_n$ and $x_n = \pi_{Q_n} Jx_0$, we have $V(Jx_0, x_n) \leq V(Jx_0, x_{n+1})$ for each $n \in N \bigcup \{0\}$. Therefore, $\{V(Jx_0, x_n)\}$ is nondecreasing. So there exists the limit of $V(Jx_0, x_n)$. By the construction of Q_n, we have that $Q_m \subset Q_n$ and $x_m = \pi_{Q_m} Jx_0 \in Q_n$ for any positive integer $m \geq n$. From property (vii) of the operator π_K, we have

$$V(Jx_n, x_m) \leq V(Jx_0, x_m) - V(Jx_0, x_n)$$

for each $n \in N \bigcup \{0\}$ and any positive integer $m \geq n$. This implies that

$$V(Jx_n, x_m) \to 0, \quad \text{as } n, m \to \infty.$$

By the definition of ϕ, we have

$$\phi(x_m, x_n) \to 0, \quad \text{as } n, m \to \infty. \tag{3.4}$$

Using lemma 2.2, we obtain

$$\|x_m - x_n\| \to 0, \quad \text{as } m, n \to \infty,$$

and hence $\{x_n\}$ is Cauchy. Therefore, there exists a point $q \in K$ such that $x_n \to q$, as $n \to \infty$. Since $x_{n+1} = \pi_{C_n \bigcap Q_n} Jx_0 \in C_n$, from the definition of C_n, we also have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n)$$

for each $n \in N \bigcup\{0\}$. Tending $n \to \infty$, we have $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. Using lemma 2.2, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

From $\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.6}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.7}$$

Since $\|Jy_n - Jx_n\| = (1-\alpha_n)\|J\pi_K(Jz_n - \beta Tz_n) - Jx_n\|$ and $\limsup_{n\to\infty} \alpha_n < 1$, we have

$$\|J\pi_K(Jz_n - \beta Tz_n) - Jx_n\| \to 0, \quad \text{as}\, n \to \infty.$$

Since J^* is also uniformly norm-to-norm continuous on bounded sets, we have

$$\|J^*J\pi_K(Jz_n - \beta Tz_n) - J^*Jx_n\| = \|\pi_K(Jz_n - \beta Tz_n) - x_n\| \to 0, \quad \text{as}\, n \to \infty. \tag{3.8}$$

Since $\{x_n\}$ is bounded, then $\{Jx_n\}$, $\{JSx_n\}$ are also bounded. Moreover, since B is a uniformly smooth Banach space, we know that B^* is a uniformly convex Banach space. Therefore, lemma 2.3 is applicable. From property(iii) of the operator V, lemma 2.3, and the definition of S, we have

$$\begin{aligned}
\phi(p, z_n) &= V(Jz_n, p) \le V(\beta_n Jx_n + (1-\beta_n)JSx_n, p)\\
&= \|\beta_n Jx_n + (1-\beta_n)JSx_n\|^2 - 2\langle \beta_n Jx_n + (1-\beta_n)JSx_n, p\rangle + \|p\|^2\\
&\le \beta_n\|Jx_n\|^2 + (1-\beta_n)\|JSx_n\|^2 - \beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|)\\
&\quad - 2\beta_n\langle Jx_n, p\rangle - 2(1-\beta_n)\langle JSx_n, p\rangle + \|p\|^2\\
&= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, Sx_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|)\\
&\le \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|)\\
&= \phi(p, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|).
\end{aligned} \tag{3.9}$$

From property (viii) of the operator V and (3.2), (3.9), we obtain

$$\begin{aligned}
\phi(p, y_n) &= V(Jy_n, p)\\
&\le \alpha_n V(Jx_n, p) + (1-\alpha_n)V(J\pi_K(Jz_n - \beta Tz_n), p)\\
&= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, \pi_K(Jz_n - \beta Tz_n))\\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n)\\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)(\phi(p, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|))\\
&= \phi(p, x_n) - (1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|) &\le \phi(p, x_n) - \phi(p, y_n)\\
&= 2\langle Jy_n - Jx_n, p\rangle + \|x_n\|^2 - \|y_n\|^2\\
&= 2\langle Jy_n - Jx_n, p\rangle + (\|x_n\| - \|y_n\|)(\|x_n\| + \|y_n\|).
\end{aligned}$$

By (3.6), (3.7) and $\limsup_{n\to\infty} \alpha_n < 1$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, we have $\lim_{n\to\infty} g(\|Jx_n - JSx_n\|) = 0$. By the property of the function g, we obtain $\lim_{n\to\infty} \|Jx_n - JSx_n\| = 0$. Since J^* is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|J^*Jx_n - J^*JSx_n\| = 0. \tag{3.10}$$

Since $x_n \to q$, we have $q \in \tilde{F}(S) = F(S)$. Moreover, $Sx_n \to q$ and $JSx_n \to Jq$. Noting properties (iii), (viii) and (ii) of the operator V, we derive that

$$\begin{aligned}\phi(x_n, z_n) &= V(Jz_n, x_n) \le V(\beta_n Jx_n + (1-\beta_n)JSx_n, x_n),\\ &\le \beta_n V(Jx_n, x_n) + (1-\beta_n)V(JSx_n, x_n),\\ &= (1-\beta_n)V(JSx_n, x_n).\end{aligned}$$

By the continuity of the operator V, we have $\lim_{n\to\infty} V(JSx_n, x_n) = V(Jq, q) = 0$ and hence $\lim_{n\to\infty}(1-\beta_n)V(JSx_n, x_n) = 0$. Therefore, $\lim_{n\to\infty} \phi(x_n, z_n) = 0$. From lemma 2.2, we have

$$\|x_n - z_n\| \to 0, \quad \text{as}\, n \to \infty. \tag{3.11}$$

Using inequalities (3.8) and (3.11), we obtain

$$\|\pi_K(Jz_n - \beta Tz_n) - z_n\| \le \|\pi_K(Jz_n - \beta Tz_n) - x_n\| + \|x_n - z_n\| \to 0. \tag{3.12}$$

Since $x_n \to q$, we have $z_n \to q$. By the continuity of the operators J, T and π_K, we have

$$\|\pi_K(Jz_n - \beta Tz_n) - \pi_K(Jq - \beta Tq)\| \to 0. \tag{3.13}$$

Noting

$$\|\pi_K(Jz_n - \beta Tz_n) - q\| \le \|\pi_K(Jz_n - \beta Tz_n) - z_n\| + \|z_n - q\| \to 0, \quad \text{as}\, n \to \infty.$$

Hence, it follows from the uniqueness of the limit that $q = \pi_K(Jq - \beta Tq)$. By Theorem 2.1, we have $q \in VI(K,T)$. Thus, $q \in VI(K,T) \bigcap F(S)$.

Finally, we show that $q = \pi_{VI(K,T)\bigcap F(S)}Jx_0$. Since $q \in VI(K,T)\bigcap F(S)$, then from property(vii) of the operator π_K, we have

$$V(J\pi_{VI(K,T)\bigcap F(S)}Jx_0, q) + V(Jx_0, \pi_{VI(K,T)\bigcap F(S)}Jx_0) \le V(Jx_0, q). \tag{3.14}$$

On the other hand, since $x_{n+1} = \pi_{C_n\bigcap Q_n}Jx_0$, and $VI(K,T)\bigcap F(S) \subset C_n \bigcap Q_n$ for each $n \in N\bigcup\{0\}$, then it follows from property(vii) of the operator π_K that

$$V(Jx_{n+1}, \pi_{VI(K,T)\bigcap F(S)}Jx_0) + V(Jx_0, x_{n+1}) \le V(Jx_0, \pi_{VI(K,T)\bigcap F(S)}Jx_0). \tag{3.15}$$

Moreover, by the continuity of the operator V, we get that

$$\lim_{n\to\infty} V(Jx_0, x_{n+1}) = V(Jx_0, q). \tag{3.16}$$

Combining (3.14), (3.15) with (3.16), we obtain that $V(Jx_0,q) = V(Jx_0, \pi_{VI(K,T)\bigcap F(S)}Jx_0)$. Therefore, it follows from the uniqueness of $\pi_{VI(K,T)\bigcap F(S)}Jx_0$ that $q = \pi_{VI(K,T)\bigcap F(S)}Jx_0$. This completes the proof. □

If $S = I$, then (3.1) reduces to the modified Mann iteration for variational inequality (1.1) and so we obtain the following result:

Corollary 3.1 *Let B be a uniformly convex and uniformly smooth Banach space. Let K be a nonempty, closed convex subset of B. Assume that T is an operator of K into B^* that satisfies conditions (2.2) and (2.3) such that $VI(K,T) \ne \emptyset$. If $T : K \to B^*$ is continuous, and the* *sequence* $\{x_n\}$ *is defined by the following modified Mann iteration*

$$\begin{cases} x_0 \in K \text{ chosen arbitrarily}, \\ y_n = J^*(\alpha_n Jx_n + (1-\alpha_n)J\pi_K(Jx_n - \beta Tx_n)), \\ C_0 = \{u \in K : \phi(u, y_0) \leq \phi(u, x_0)\}, \\ C_n = \{u \in C_{n-1} \bigcap Q_{n-1} : \phi(u, y_n) \leq \phi(u, x_n)\}, \\ Q_0 = K, \\ Q_n = \{u \in Q_{n-1} \bigcap C_{n-1} : \langle Jx_0 - Jx_n, x_n - u\rangle \geq 0\}, \\ x_{n+1} = \pi_{C_n \bigcap Q_n} Jx_0, \end{cases} \tag{3.17}$$

where $\{\alpha_n\}$ *satisfies:*

$$0 \leq \alpha_n < 1, \quad and \quad \limsup_{n\to\infty} \alpha_n < 1.$$

Then the sequence $\{x_n\}$ *converges strongly to* $\pi_{VI(K,T)}Jx_0$, *where* $\pi_{VI(K,T)}$ *is the generalized projection from* B^* *onto* $VI(K,T)$.

Proof Taking $S = I$ in Theorem 3.1, by $x_n \in K$ and property(iv) of the operator π_K, we have $z_n = \pi_K Jx_n = x_n$. Thus, we can obtain the desired conclusion. □

Remark 3.1 Corollary 3.1 improves theorem 3.3 of [9] and Theorem 3.1 of [8] in the following senses:

(1) the condition in theorem 3.3 of [9] that $J - \beta T : K \to B^*$ is compact is removed, we only require that $T : K \to B^*$ is continuous;
(2) we obtain that the convergence point of $\{x_n\}$ is $\pi_{VI(K,T)}Jx_0$, which is more concrete than related conclusions of [8] and [9].

If $K = B$, we obtain the following result:

Corollary 3.2 *Let* B *be a uniformly convex and uniformly smooth Banach space. Let* T *be an operator of* B *into* B^* *that satisfy the following conditions: there exists a positive number* β *such that*

$$\langle Tx, J^*(Jx - \beta Tx)\rangle \geq 0, \quad \forall x \in B$$

and

$$\langle Tx, y\rangle \leq 0, \quad \forall x \in B, y \in T^{-1}0,$$

where $T^{-1}0 = \{u \in B : Tu = 0\}$. *Suppose that* $S : B \to B$ *is a relatively weak nonexpansive mapping with* $T^{-1}0 \bigcap F(S) \neq \emptyset$. *If* $T : B \to B^*$ *is continuous, then the sequence* $\{x_n\}$ *defined by the following iteration process:*

$$\begin{cases} x_0 \in B \text{ chosen arbitrarily}, \\ z_n = J^*(\beta_n Jx_n + (1-\beta_n)JSx_n) \\ y_n = J^*(\alpha_n Jx_n + (1-\alpha_n)(Jz_n - \beta Tz_n)), \\ C_0 = \{u \in B : \phi(u, y_0) \leq \phi(u, x_0)\}, \\ C_n = \{u \in C_{n-1} \bigcap Q_{n-1} : \phi(u, y_n) \leq \phi(u, x_n)\}, \\ Q_0 = B, \\ Q_n = \{u \in Q_{n-1} \bigcap C_{n-1} : \langle Jx_0 - Jx_n, x_n - u\rangle \geq 0\}, \\ x_{n+1} = \pi_{C_n \bigcap Q_n} Jx_0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ *satisfy:*

$$0 \leq \alpha_n < 1, \quad and \quad \limsup_{n\to\infty} \alpha_n < 1; 0 < \beta_n < 1 \quad and \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0,$$

converges strongly to $\pi_{T^{-1}0 \bigcap F(S)} J x_0$.

Proof Taking $K = B$ in Theorem 3.1, by lemma 2.6 and Theorem 2.1, we have $\pi_B = J^*$ and $VI(B, T) = T^{-1}0$. Therefore, it is easy to obtain the desired result by Theorem 3.1. □

References

1. Alber, Ya.: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos, A. (ed.) Theory and Applications of Nonlinear Operators of Monotonic and Accretive Type, pp. 15–50. Marcel Dekker, New York (1996)
2. Alber, Ya., GuerreDelabriere, S.: On the projection methods for fixed point problems. Analysis **21**, 17–39 (2001)
3. Alber, Ya., Notik, A.: On some estimates for projection operator in Banach space. Comm. Appl. Nonlinear Anal. **2**, 47–56 (1995)
4. Alber, Ya., Reich, S.: An iterative method for solving a class of nonlinear operator equations in Banach spaces. Panamer. Math. J. **4**, 39–54 (1994)
5. Chang, S.-s.: On Chidumes open questions and approximate solutions of multivalued strongly accretive mapping in Banach spaces. J. Math. Anal. Appl. **216**, 94–111 (1997)
6. Chidume, C.E., Li, J.: Projection methods for approximating fixed points of Lipschitz suppressive operators. Panamer. Math. J. **15**, 29–40 (2005)
7. Chidume, C.E.: Iterative solutions of nonlinear equations in smooth Banach spaces. Nonlinear Anal. **26**, 1823–1834 (1996)
8. Li, J.: On the existence of solutions of variational inequalities in Banach spaces. J. Math. Anal. Appl. **295**, 115–126 (2004)
9. Jianghua, F.: A Mann type iterative scheme for variational inequalities in noncompact subsets of Banach spaces. J. Math. Anal. Appl. **337**, 1041–1047 (2008)
10. Kohasaka, F., Takahashi, W.: Strong convergence of an iterative sequence for maximal monotone operators in Banach spaces. Abstr. Appl. Anal. **2004**(3), 239–249 (2004)
11. Li, J.: The generalized projection operator on reflexive Banach spaces and its applications. J. Math. Anal. Appl. **306**, 55–71 (2005)
12. Butanriu, D., Reich, S., Zaslavski, A.J.: Asymtotic behavior of relatively nonexpansive opera- tors in Banach spaces. J. Appl. Anal. **7**, 151–174 (2001)
13. Butanriu, D., Reich, S., Zaslavski, A.J.: Weakly convergence of orbits of nonlinear operators in reflexive Banach spaces. Numer. Funct. Anal. Optim. **24**, 489–508 (2003)
14. Matsushita, S.-y., Takahashi, W.: A strong convergence theorem for relatively nonexpansive mappings in a Banach space. J. Approx. Theory **134**, 257–266 (2005)
15. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127–1138 (1991)